Dirac-Like Equation for the Photon

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A Dirac-like equation for the photon is considered by using a four-by-two matrix wave function. Conservation laws and transformation properties under Lorentz transformations are considered.

Maxwell's equations were recently written in the Dirac form by Sallhofer [1] by using a wave function whose terms are the components of the electric and magnetic fields. Actually, this problem has been attracting the attention of several authors [2–6]. Some of them assume that the fourth component of the wave function is null, an assumption that is connected with the transversality of the electromagnetic wave. It is possible to show, however, that the additional assumption can be avoided by conveniently modifying the wave function to a form that has peculiar transformation properties under Lorentz transformations [6].

It is the aim of the present note to show that Maxwell's equations can be written in the Dirac form without the additional condition even if use is made of a wave function different from the one used by Edmonds [6]. In this way Sallhofer's wave function is generalized.

Let us consider Maxwell's equations in the form [7]

$$\begin{aligned}&\operatorname{rot}\boldsymbol{E}+\dot{\boldsymbol{H}}=0, \quad \operatorname{div}\boldsymbol{H}=0\,,\\ &\operatorname{rot}\boldsymbol{H}-\dot{\boldsymbol{E}}=4\pi\boldsymbol{j}\,, \quad \operatorname{div}\boldsymbol{E}=4\pi\varrho\,,\end{aligned} \tag{1}$$

($\hbar = c = 1$).

If we use the relation

$$(\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})(\boldsymbol{\sigma}\cdot\boldsymbol{A})=\boldsymbol{\nabla}\cdot\boldsymbol{A}+i\,\boldsymbol{\sigma}\cdot(\boldsymbol{\nabla}\wedge\boldsymbol{A}) \tag{2}$$

we can write (1) in the form

$$\left[\begin{pmatrix}\boldsymbol{\sigma} & 0\\ 0 & \boldsymbol{\sigma}\end{pmatrix}\cdot\boldsymbol{\nabla}+i\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\frac{\partial}{\partial t}\right]\begin{bmatrix}\boldsymbol{\sigma}\cdot\boldsymbol{E}\\ \boldsymbol{\sigma}\cdot\boldsymbol{H}\end{bmatrix} = 4\pi\begin{bmatrix}\varrho I\\ i\,\boldsymbol{\sigma}\cdot\boldsymbol{j}\end{bmatrix}. \tag{3}$$

If we multiply both members of (3) by

$$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

we obtain

$$\gamma_\mu\,\partial_\mu\,\psi = J\,, \tag{4}$$

where γ_μ are the Dirac matrices. ψ is a wave function with two columns and four rows. Its components depend on E and H. The well-known solutions

$$(1\pm\gamma_5)\,\psi$$

of (4) for the vacuum separate the dependence of ψ on $\boldsymbol{E}$ and $\boldsymbol{H}$. The Hamiltonian is also easy to be obtained. If Maxwell's equations were used it is easy to see that the wave function ψ is an eigenfunction of the Hamiltonian in the stationary state. In this case the expectation values of the Hamiltonian are the eigenvalues of this operator. This result is not important for us, however, because we are considering a classical theory from a different point of view. It is important to observe that the two columns of ψ are also solutions of (3). One of these columns is the wave function indicated by Sallhofer [1]. It should be observed that these solutions ψ_1 and ψ_2 are not connected by an unitary matrix.

The conservation law

$$(1/2)\,\frac{\partial}{\partial t}\,(E^2+H^2)+\frac{\partial}{\partial x^k}\,[\boldsymbol{E}\wedge\boldsymbol{H}]_k=0 \tag{5}$$

results from (3), in vacuum, if we take the following form for the adjoint of (3):

$$\bar{\psi}\,\gamma_\mu\,\partial_\mu=0$$

where

$$\bar{\psi}=\psi^{\mathrm{HC}}\,\gamma_4\,.$$

If we use the Eq. (3), for vacuum, and its adjoint we obtain the conservation law

$$\partial_\mu\, j_\mu = 0 \tag{6}$$

where

$$j_\mu=\psi^{\mathrm{HC}}\,\gamma_4\,\gamma_\mu\,\psi\,. \tag{7}$$

The relation (5) is the fourth component of a tensor equation. Therefore, in (6) j_μ is not a fourvector, so that we should expect a peculiar behavior of the Eq. (3) under Lorentz transformations. The transformation of the wave function is in this case

$$\psi_{1,\,2}'=LS\,\psi_{1,\,2}$$

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where, as usual,

$$a_{\mu\nu}\,\gamma_\nu = L^{+1}\,\gamma_\mu\,L^{-1}$$

and the matrix S occurs because components of $\boldsymbol{E}$ and $\boldsymbol{H}$ appears in $\psi_{1,2}$. The matrix S is not unitary so that j_μ is not a four-vector. Let us observe that the terms of the wave function ψ are not scalars. These problems are connected with the fact that the relations (1) have not a covariant form. Nevertheless, we can see from (3) that the product of the space inversion by the time reversion is the identity.

If we consider the complete Eq. (3) and apply to its left the matrix

$$\begin{bmatrix} I\,\dfrac{\partial}{\partial t} \\ -\,i\,\boldsymbol{\sigma}\cdot\boldsymbol{\nabla} \end{bmatrix}^T,$$

where T is for transposed, we obtain the usual charge-current conservation law. This deduction is not elegant, however, from the mathematical point of view, because we use at the same time the relations (1) and (3).

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